Find the Mistake - Answers

All of the following problems contain a mistake. Identify and correct each one.

1. **Section 2.2**: The negation of "$1 < a < 5$" is "$1 \geq a \geq 5$."

   Answer: A statement of the form "$1 < a < 5$" is an *and* statement. Thus, by De Morgan's law, its negation is an *or* statement. The correct negation is $1 \geq a$ or $a \geq 5$.

2. **Section 2.2**: "P only if Q" means "if Q then P."

   Answer: "P only if Q" means that the only way P can occur is for Q to occur. This means that if Q does not occur, then P cannot occur, or, equivalently, "if P occurs then Q must have occurred," i.e., "if P then Q."

3. **Section 3.2**

   (a) The negation of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "For all real numbers x, if $x > 2$ then $x^2 \leq 4$."

   (b) The negation of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "There exist real numbers x such that if $x > 2$ then $x^2 \leq 4$."

   (c) The negation of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "There exists a real number x such that $x > 2$ and $x^2 < 4$."

   Answer to a, b, and c: The negation of a "For all" statement is a "There exists" statement, the negation of "if p then q" is "p and not q," and the negation of "$x^2 > 4$" is "$x^2 \leq 4$." The correct negation in all three cases is "There exists a real number x such that $x > 2$ and $x^2 \leq 4$."

4. **Section 3.2**: The contrapositive of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "For all real numbers x, if $x \leq 2$ then $x^2 \leq 4$."

   Answer: The contrapositive of "if p then q" is "if not q then not p." In this case p is $x > 2$ and q is $x^2 > 4$. Thus the correct answer is "For all real numbers x, if $x^2 \leq 4$ then $x \leq 2$."

5. **Section 3.3**: Statement: $\exists$ a real number x such that $\forall$ real numbers y, $x + y = 0$. Proposed negation: $\forall$ real numbers x, if y is a real number then $x + y \neq 0$.

   Answer: The proposed negation began correctly with "$\forall$ real numbers x," but the continuation should be the existential statement " $\exists$ a real number y such that $x + y \neq 0$."

6. **Section 4.1**: A person is asked to prove that the square of any odd integer is odd. Toward the end of a proof the person writes: "Therefore $n^2 = 2k + 1$, which is the definition of odd."

   Answer: For an integer to be odd means that it equals 2 times some integer plus 1.So it is not correct to say that "$2k + 1$ *is* the definition of odd." The person should have written: "Therefore $n^2 = 2k + 1$, where k is an integer, and so n^2 is odd by definition of odd."

7. **Section 4.1**: *Prove:* The square of any even integer is even.

   Beginning of proof: Suppose that r is any integer. Then if m is any even integer, $m = 2r. \ldots$

   Answer: To prove that the square of any even integer is even, you must start by supposing you have a *[particular but arbitrarily chosen]* even integer. By using the definition of even, you can *deduce* what the even integer must look like, namely that it must equal $2 \cdot$(some integer). A correct proof would start with an even integer m and deduce the existence of an integer r such that $m = 2r$. This "proof" has it backwards.

8. **Section 4.1**: *Prove directly from the definition of even:* For all even integers n, $(-1)^n = 1$.

   Beginning of proof: Suppose n is any even integer. Then $n = 2r$ for some integer r. By substitution, $(-1)^n = (-1)^{2r} = 1$ because $2r$ is even....

   Answer: By claiming that $(-1)^{2r} = 1$, this "proof" assumes what is to be proved, namely that (-1) raised to an even power equals 1.

9. **Section 4.1**: *Prove directly from the definition of even:* For all even integers n, $(-1)^n = 1$.

   Beginning of proof: Suppose n is any even integer. Then $n = 2r$ for some integer r. By substitution, $(-1)^{2r} = ((-1)^2)^r$...

   Answer: The fact that $(-1)^{2r} = ((-1)^2)^r$ follows from a property of exponents; it is not true "by substitution." When you write "by substitution," you have to include the original variable in the equation that you write. Thus the following would be correct:
   Prove directly from the definition of even: For all even integers n, $(-1)^n = 1$.

   Beginning of proof: Suppose n is any even integer. Then $n = 2r$ for some integer r, and so

   $$\begin{aligned} (-1)^n &= (-1)^{2r} && \text{by substitution} \\ &= ((-1)^2)^r && \text{by a property of exponents...} \end{aligned}$$

10. **Section 4.3**: *Prove:* For all integers a and b, if a and b are divisible by 3 then $a+b$ is divisible by 3.

    Beginning of proof: Suppose that for all integers a and b, if a and b are divisible by 3 then $a + b$ is divisible by 3. By definition of divisibility, ...

    Answer: This proof begins by assuming exactly what is to be proved. If one assumes what is to be proved, there is nothing left to do!

11. **Section 4.3**: *Prove:* For all integers a, if 3 divides a, then 3 divides a^2.

    Beginning of proof: Suppose a is any integer such that 3 divides a. Then $a = 3k$ for any integer k....

    Answer: It is incorrect to say that "$a = 3k$ for any integer k" because k cannot be just "any" integer; in fact, the only integer that k can be is $k = a/3$. The correct thing to say is, "Then $a = 3k$ for some integer k."

12. **Section 4.3**: *Prove:* For all integers a, if $a = 3b+1$ for some integer b, then $a^2 - 1$ is divisible by 3.

    Beginning of proof: Let a be any integer such that $a = 3b + 1$ for some integer b. We will prove that $a^2 - 1$ is divisible by 3. This means that $a^2 - 1 = 3q$ for some integer q. Then $(3b + 1)^2 - 1 = 3q$, and, since q is an integer, by definition of divisibility, $a^2 - 1$ is divisible by 3....

    Answer: This "proof" assumes something equivalent to what is to be proved. After stating "We will prove that $a^2 - 1$ is divisible by 3" it is correct to state that ."This means that $a^2 - 1 = 3q$ for some integer q." However, the following sentence assumes that the integer q has been shown to exist, which is not the case.

13. **Section 4.4**: *Prove:* For all integers a, $a^2 - 2$ is not divisible by 3.

    Beginning of proof: Suppose a is any integer. By the quotient-remainder theorem with divisor $d = 3$, there exist unique integers q and r such that $a = 3q + r$, where $0 < r \leq 3$.

    Answer: The inequality is incorrect; it should be $0 \leq r < 3$.

14. **Section 4.6**: *Prove by contradiction:* The product of any irrational number and any rational number is irrational.

Beginning of proof: Suppose not. That is, suppose the product of any irrational number and any rational number is rational.

Answer: A proof by contradiction start with the negations of the statement to be proved. In this case, the statement to be proved is universal, and so its negation is existential. However, this proposed proof begins with a universal statement. A correct way to begin the proof is the following:

Beginning of proof: Suppose not. That is, suppose there exists an irrational number and a rational number whose product is rational.

15. **Section 4.6**: The negation of "n is not divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$" is "n is divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$."

    Answer: Consider negating the statement "He does not have any money." The negation is not "He does have any money," it is "He does have some money." Similarly, the negation of "n is not divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$" is not "n is divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$." It is "n is divisible by *some* prime number greater than 1 and less than or equal to $\sqrt{n}$," or "There exists a prime number greater than 1 and less than or equal to $\sqrt{n}$ that divides n."

16. **Section 5.2**: The equation $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ is true for $n = 1$ because $1 + 2 + 3 + \cdots + 1 = \frac{1(1+1)}{2}$ is true.

    Answer: When $n = 1$, the expression $1+2+3+\cdots+n = 1$; it does not equal $1+2+3+\cdots+1$.

17. **Section 5.2**: The equation $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ is true for $n = 1$ because

    $$1 = \frac{1(1+1)}{2} \Rightarrow 1 = \frac{2}{2} \Rightarrow 1 = 1.$$

    Answer: A false statement can imply a true conclusion. So deducing a true conclusion from a statement is not a valid way to prove that the statement is true.

18. **Section 5.2**: *Prove by mathematical induction:* For all integers $n \geq 1$,

    $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

    Beginning of proof: Let the property $P(n)$ be

    $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \text{ for all integers } n \geq 1....$$

    Answer: The job of a proof by mathematical induction is to prove that a given property is true for all integers greater than or equal to a given integer. In this example, the property $P(n)$ is simply the equation

    $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

    and the proof by mathematical induction establishes that $P(n)$ is true for all integers $n \geq 1$. The mistake is including the words "for all integers $n \geq 1$" as part of $P(n)$ because these words make $P(n)$ identical with what is to be proved.

19. **Section 6.1**: Given sets A and B, to show that A is a subset of B, we must show that there is an element x such that x is in A and x is in B.

    Answer: This answer implies that for A to be a subset of B, it is enough for there to be a single element that is in both sets. But this is false. For instance, if $A = \{1, 2\}$ and $B = \{2, 3\}$, then 2 is in both A and B, but A is not a subset of B because 1 is in A and 1 is not in B. In fact, for A to be a subset of B means that for all x, ***if*** x is in A ***then*** x must be in B.

20. **Section 6.1**: Given sets A and B, to show that A is a subset of B, we must show that for all x, x is in A and x is in B.

    Answer: There are two problems with this answer. One is that it implies that A and B are identical sets, whereas for A to be a subset of B it is possible for B to contain elements that are not in A. In addition, because no domain is specified for x, it appears to say that everything in the universe is in both A and B, which is not the case for most sets A and B.

21. **Section 7.2**: To prove that $F\colon A \to B$ is one-to-one, assume that if $F(x_1) = F(x_2)$ then $x_1 = x_2$.

    Answer: Assuming that "if $F(x_1) = F(x_2)$ then $x_1 = x_2$" is essentially the same as assuming that F is one-to-one. In other words, it essentially assumes what needs to be proved.

22. **Section 7.2**: To prove that $F\colon A \to B$ is one-to-one, we must show that for all x_1 and x_2 in A, $F(x_1) = F(x_2)$ and $x_1 = x_2$.

    Answer: This statement implies that for all x_1 and x_2 in A, $x_1 = x_2$. In other words, it implies that there is only one element in A, which is very seldom the case.

23. **Section 8.2**: Define a relation R on the set of all integers by $a\,R\,b$ if, and only if, $ab > 0$. To show that R is symmetric, assume that for all integers a and b, $a\,R\,b$. We will show that $b\,R\,a$.

    Answer: The problem with these statements is that saying "assume that for all integers a and b, $a\,R\,b$" is equivalent to saying that every integer is related to every other integer by R. This is not the case. For instance, -1 is not related to 1 because $(-1) \cdot 1 = -1$ and $-1 \not> 0$.