Review Guide: Chapter 5

Sequences and Summations

- What is a method to help find an explicit formula for a sequence whose first few terms are given (provided a nice explicit formula exists!)? *(p. 229)*
- What is the expanded form for a sum that is given in summation notation? *(p. 231)*
- What is the summation notation for a sum that is given in expanded form? *(p. 231)*
- How do you evaluate $a_1 + a_2 + a_3 + \cdots a_n$ when n is small? *(p. 232)*
- What does it mean to "separate off the final term of a summation"? *(p. 232)*
- What is the product notation? *(p. 233)*
- What are some properties of summations and products? *(p. 234)*
- How do you transform a summation by making a change of variable? *(p. 235)*
- What is factorial notation? *(p. 237)*
- What is the n choose r notation? *(p. 238)*
- What is an algorithm for converting from base 10 to base 2? *(p. 241)*

Mathematical Induction

- What do you show in the basis step and what do you show in the inductive step when you use (ordinary) mathematical induction to prove that a property involving an integer n is true for all integers greater than or equal to some initial integer? *(p. 247)*
- What is the inductive hypothesis in a proof by (ordinary) mathematical induction? *(p. 247)*
- Are you able to use (ordinary) mathematical induction to construct proofs involving various kinds of statements such as formulas, divisibility properties, and inequalities? *(pp. 247, 249, 253, 259, 263, 265)*
- Are you able to apply the formula for the sum of the first n positive integers? *(p. 251)*
- Are you able to apply the formula for the sum of the successive powers of a number, starting with the zeroth power? *(p. 255)*

Strong Mathematical Induction and The Well-Ordering Principle for the Integers

- What do you show in the basis step and what do you show in the inductive step when you use strong mathematical induction to prove that a property involving an integer n is true for all integers greater than or equal to some initial integer? *(p. 268)*
- What is the inductive hypothesis in a proof by strong mathematical induction? *(p. 268)*
- Are you able to use strong mathematical induction to construct proofs of various statements? *(pp. 269-274)*
- What is the well-ordering principle for the integers? *(p. 275)*
- Are you able to use the well-ordering principle for the integers to prove statements, such as the existence part of the quotient-remainder theorem? *(p. 276)*
- How are ordinary mathematical induction, strong mathematical induction, and the well-ordering principle for the integers related? *(p. 275 and exercises 31 and 32 on p. 279)*

Algorithm Correctness

- What are the pre-condition and the post-condition for an algorithm? *(p. 280)*

- What does it mean for a loop to be correct with respect to its pre- and post-conditions? *(p. 281)*
- What is a loop invariant? *(p. 282)*
- How do you use the loop invariant theorem to prove that a loop is correct with respect to its pre- and post-conditions? *(pp. 283-288)*

Recursion

- What is an explicit formula for a sequence? *(p. 290)*
- What does it mean to define a sequence recursively? *(p. 290)*
- What is a recurrence relation with initial conditions? *(p. 290)*
- How do you compute terms of a recursively defined sequence? *(p. 290)*
- Can different sequences satisfy the same recurrence relation? *(p. 291)*
- What is the "recursive paradigm"? *(p. 293)*
- How do you develop recurrence relations for sequences that are variations of the towers of Hanoi sequence? *(p. 294 and the solutions on p. A-42 to exercises 17 and 20 in Section 5.7)*
- How do you develop a recurrence relations for the Fibonacci sequence? *(p. 297)*
- How do you develop recurrence relations for sequences that involve compound interest? *(p. 298-299)*

Solving Recurrence Relations

- What is the method of iteration for solving a recurrence relation? *(p. 305)*
- What is an arithmetic sequence? *(p. 307)*
- What is a geometric sequence? *(p. 308)*
- How do you use the formula for the sum of the first n integers and the formula for the sum of the first n powers of a real number r to simplify the answers you obtain when you solve recurrence relations? *(pp. 309-310)*
- How is mathematical induction used to check that the solution to a recurrence relation is correct? *(p. 312-314)*
- What is a second-order linear homogeneous recurrence relation with constant coefficients? *(p. 317)*
- What is the characteristic equation for a second-order linear homogeneous recurrence relation with constant coefficients? *(p. 319)*
- What is the distinct-roots theorem? If the characteristic equation of a relation has two distinct roots, how do you solve the relation? *(p. 321)*
- What is the single-root theorem? If the characteristic equation of a relation has a single root, how do you solve the relation? *(p. 325)*

General Recursive Definitions

- When a set is defined recursively, what are the three parts of the definition? *(p. 328)*
- Given a recursive definition for a set, how can you tell that a given element is in the set? *(p. 328-329)*
- What is structural induction? *(p. 331)*
- Given a recursive definition for a set, is there a way to tell that a given element is not in the set? *(solution for exercise 14a on p. A-49)*
- What is a recursive function? *(p. 332)*

Formats for Proving Formulas by Mathematical Induction

When using mathematical induction to prove a formula, students are sometimes tempted to present their proofs in a way that assumes what is to be proved. There are several formats you can use, besides the one shown most frequently in the textbook, to avoid this fallacy. A crucial point is this:

> If you are hoping to prove that an equation is true but you haven't yet done so, either preface it with the words "We must show that" or put a question mark above the equal sign.

Format 1 (the format used most often in the textbook for the inductive step): Start with the left-hand side (LHS) of the equation to be proved and successively transform it using definitions, known facts from basic algebra, and (for the inductive step) the inductive hypothesis until you obtain the right-hand side (RHS) of the equation.

Format 2 (the format used most often in the textbook for the basis step): Transform the LHS and the RHS of the equation to be proved *independently*, one after the other, until both sides are shown to equal the same expression. Because two quantities equal to the same quantity are equal to each other, you can conclude that the two sides of the equation are equal to each other.

Format 3: This format is just like Format 2 except that the computations are done in parallel. But in order to avoid the fallacy of assuming what is to be proved, do NOT put an equal sign between the two sides of the equation until the very last step. Separate the two sides of the equation with a vertical line.

Format 4: This format is just like Format 3 except that the two sides of the equation are separated by an equal sign with a question mark on top: $\stackrel{?}{=}$

Format 5: Start by writing something like "We must show that" and the equation you want to prove true. In successive steps, indicate that this equation is true if, and only if, ($\Leftrightarrow$) various other equations are true. But be sure that both the directions of your "if and only if" claims are correct. In other words, be sure that the $\Leftarrow$ direction is just as true as the $\Rightarrow$ direction. If you finally get down to an equation that is known to be true, then because each subsequent equation is true *if, and only if*, the previous equation is true, you will have shown that the original equation is true.

Example: Let the property $P(n)$ be the equation

$$\boxed{1 + 3 + 5 + \cdots + (2n - 1) = n^2} \leftarrow P(n).$$

Proof that $P(1)$ is true:

Solution (Format 2):
When $n = 1$, the LHS of $P(1)$ equals 1, and the RHS equals 1^2 which also equals 1. So $P(1)$ is true.

Proof that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Solution (Format 2):
Suppose that k is any integer with $k \geq 1$ such that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1 + 3 + 5 + \cdots + (2k + 1) = (k + 1)^2$.

Now the LHS of $P(k+1)$ is

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &= 1+3+5+\cdots+(2k-1)+(2k+1) \\ &\qquad \text{by making the next-to-last term explicit} \\ &= k^2+(2k+1) \quad \text{by inductive hypothesis.} \end{aligned}$$

And the RHS of $P(k+1)$ is

$$(k+1)^2 = k^2+2k+1 \quad \text{by basic algebra.}$$

So the left-hand and right-hand sides of $P(k+1)$ equal the same quantity, and thus and thus $P(k+1)$ is true *[as was to be shown].*

Solution (Format 3):
Suppose that k is any integer with $k \geq 1$ such that $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1+3+5+\cdots+(2k+1) = (k+1)^2$.
Consider the left-hand and right-hand sides of $P(k+1)$:

$1+3+5+\cdots+(2k+1)$	$(k+1)^2$
$= 1+3+5+\cdots+(2k-1)+(2k+1)$	
by making the next-to-last term explicit	
$= k^2+(2k+1)$	
by inductive hypothesis	
$= k^2+2k+1$	$= k^2+2k+1$
by basic algebra	by basic algebra

So the left-hand and right-hand sides of $P(k+1)$ equal the same quantity, and thus and thus $P(k+1)$ is true *[as was to be shown].*

Solution (Format 4):
Suppose that k is any integer with $k \geq 1$ such that $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1+3+5+\cdots+(2k+1) = (k+1)^2$.
Consider the left-hand and right-hand sides of $P(k+1)$:

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &\stackrel{?}{=} (k+1)^2 \\ 1+3+5+\cdots+(2k-1)+(2k+1) &\stackrel{?}{=} k^2+2k+1 \\ \text{by making the next-to-last term explicit} & \qquad \text{by basic algebra} \\ k^2+(2k+1) &\stackrel{?}{=} k^2+2k+1 \\ \text{by inductive hypothesis} & \\ k^2+2k+1 &= k^2+2k+1 \\ \text{by basic algebra} & \end{aligned}$$

So the left-hand and right-hand sides of $P(k+1)$ equal the same quantity, and thus $P(k+1)$ is true *[as was to be shown].*

Solution (Format 5):
Suppose that k is any integer with $k \geq 1$ such that $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1+3+5+\cdots+(2k+1) = (k+1)^2$.
But $P(k+1)$ is true if, and only if, ($\Leftrightarrow$)

$$\begin{aligned} & 1+3+5+\cdots+(2k-1)+(2k+1) = (k+1)^2 && \text{by making the next-to-last term explicit} \\ \Leftrightarrow\ & k^2+(2k+1) = (k+1)^2 && \text{by inductive hypothesis} \\ \Leftrightarrow\ & k^2+2k+1 = (k+1)^2 && \end{aligned}$$

which is true by basic algebra. Thus $P(k+1)$ is true *[as was to be shown].*