Formats for Proving Formulas by Mathematical Induction

When using mathematical induction to prove a formula, students are sometimes tempted to present their proofs in a way that assumes what is to be proved. There are several formats you can use, besides the one shown most frequently in the textbook, to avoid this fallacy. A crucial point is this:

If you are hoping to prove that an equation is true but you haven't yet done so, either preface it with the words "We must show that" or put a question mark above the equal sign.

Format 1 (the format used most often in the textbook for the inductive step): Start with the left-hand side (LHS) of the equation to be proved and successively transform it using definitions, known facts from basic algebra, and (for the inductive step) the inductive hypothesis until you obtain the right-hand side (RHS) of the equation.

Format 2 (the format used most often in the textbook for the basis step): Transform the LHS and the RHS of the equation to be proved *independently*, one after the other, until both sides are shown to equal the same expression. Because two quantities equal to the same quantity are equal to each other, you can conclude that the two sides of the equation are equal to each other.

Format 3: This format is just like Format 2 except that the computations are done in parallel. But in order to avoid the fallacy of assuming what is to be proved, do NOT put an equal sign between the two sides of the equation until the very last step. Separate the two sides of the equation with a vertical line.

Format 4: This format is just like Format 3 except that the two sides of the equation are separated by an equal sign with a question mark on top: $\stackrel{?}{=}$

Format 5: Start by writing something like "We must show that" and the equation you want to prove true. In successive steps, indicate that this equation is true if, and only if, ($\Leftrightarrow$) various other equations are true. But be sure that both the directions of your "if and only if" claims are correct. In other words, be sure that the $\Leftarrow$ direction is just as true as the $\Rightarrow$ direction. If you finally get down to an equation that is known to be true, then because each subsequent equation is true *if, and only if*, the previous equation is true, you will have shown that the original equation is true.

Example: Let the property $P(n)$ be the equation

$$\boxed{1 + 3 + 5 + \cdots + (2n - 1) = n^2} \leftarrow P(n).$$

Proof that $P(1)$ is true:

Solution (Format 2):
When $n = 1$, the LHS of $P(1)$ equals 1, and the RHS equals 1^2 which also equals 1. So $P(1)$ is true.

Proof that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Solution (Format 2):
Suppose that k is any integer with $k \geq 1$ such that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k + 1)$ is true, where $P(k + 1)$ is the equation $1 + 3 + 5 + \cdots + (2k + 1) = (k + 1)^2$.

Now the LHS of $P(k+1)$ is

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &= 1+3+5+\cdots+(2k-1)+(2k+1) \\ &\qquad \text{by making the next-to-last term explicit} \\ &= k^2+(2k+1) \quad \text{by inductive hypothesis.} \end{aligned}$$

And the RHS of $P(k+1)$ is

$$(k+1)^2 = k^2+2k+1 \quad \text{by basic algebra.}$$

So the left-hand and right-hand sides of $P(k+1)$ equal the same quantity, and thus and thus $P(k+1)$ is true *[as was to be shown]*.

Solution (Format 3):
Suppose that k is any integer with $k \geq 1$ such that $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1+3+5+\cdots+(2k+1) = (k+1)^2$.
Consider the left-hand and right-hand sides of $P(k+1)$:

$$\begin{array}{l|l} 1+3+5+\cdots+(2k+1) & (k+1)^2 \\ =1+3+5+\cdots+(2k-1)+(2k+1) & \\ \qquad \text{by making the next-to-last term explicit} & \\ =k^2+(2k+1) & \\ \qquad \text{by inductive hypothesis} & \\ =k^2+2k+1 & =k^2+2k+1 \\ \qquad \text{by basic algebra} & \qquad \text{by basic algebra} \end{array}$$

So the left-hand and right-hand sides of $P(k+1)$ equal the same quantity, and thus and thus $P(k+1)$ is true *[as was to be shown]*.

Solution (Format 4):
Suppose that k is any integer with $k \geq 1$ such that $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1+3+5+\cdots+(2k+1) = (k+1)^2$.
Consider the left-hand and right-hand sides of $P(k+1)$:

$$\begin{array}{lcl} 1+3+5+\cdots+(2k+1) & \stackrel{?}{=} & (k+1)^2 \\ 1+3+5+\cdots+(2k-1)+(2k+1) & \stackrel{?}{=} & k^2+2k+1 \\ \qquad \text{by making the next-to-last term explicit} & & \qquad \text{by basic algebra} \\ k^2+(2k+1) & \stackrel{?}{=} & k^2+2k+1 \\ \qquad \text{by inductive hypothesis} & & \\ k^2+2k+1 & = & k^2+2k+1 \\ \qquad \text{by basic algebra} & & \end{array}$$

So the left-hand and right-hand sides of $P(k+1)$ equal the same quantity, and thus $P(k+1)$ is true *[as was to be shown]*.

Solution (Format 5):
Suppose that k is any integer with $k \geq 1$ such that $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1+3+5+\cdots+(2k+1) = (k+1)^2$.
But $P(k+1)$ is true if, and only if, ($\Leftrightarrow$)

$$\begin{array}{lrcll} & 1+3+5+\cdots+(2k-1)+(2k+1) & = & (k+1)^2 & \text{by making the next-to-last term explicit} \\ \Leftrightarrow & k^2+(2k+1) & = & (k+1)^2 & \text{by inductive hypothesis} \\ \Leftrightarrow & k^2+2k+1 & = & (k+1)^2 & \end{array}$$

which is true by basic algebra. Thus $P(k+1)$ is true *[as was to be shown]*.